Income Inequality and Growth: Calibration and Simulation for the Kenyan Economy

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Gilbert Mbara

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List of abbreviations and acronyms

ABH	Aiyagari–Bewley–Huggett
CBS	Central Bureau of Statistics
CDF	Cumulative Distribution Function
CRRA	Constant Relative Risk Aversion
DINA	Distributional National Accounts
DSGE	Dynamic Stochastic General Equilibrium
FP	Fokker-Planck
GDP	Gross Domestic Product
HJB	Hamilton-Bellman-Jacobi
IID	Integrated Inequality Dataset
IMF	International Monetary Fund
KNBS	Kenya National Bureau of Statistics
OECD	Organisation for Economic Co-operation and Development
PAYE	Pay As You Earn
PDEs	Partial Differential Equations
PIP	Poverty Inequality Platform
SAM	Social Accounting Matrix
SSA	Sub-Saharan Africa
UNDP	United Nations Development Programme
UNESCO	United Nations Educational, Scientific and Cultural Organization
WID	World Inequality Database

Abstract

We investigate the notable decline in wealth and income inequality in Kenya over the 10-year period between 2005 and 2015. Using a calibrated continuous time heterogeneous agent model, we attribute up to 92% of the variation in top wealth inequality to a persistent but slow increase in the return to capital, a low risk free rate, and rising “effective” income tax rates. Our study suggests that a macroeconomic environment characterized by low risk-free interest rates anchored by low debt-to-fiscal revenue ratios are key to reducing both wealth and income inequality.

1. Introduction

Over time and across countries, wealth and income distributions appear to be highly right skewed: a small fraction of the population owns a large share of the economy's wealth and income. For example, in the United States, the top 0.1% richest individuals own up to 20% of national wealth (Saez & Zucman, 2016). Wealth and income inequality in Kenya is similar: in 2020, the top 1% richest individuals owned 28% and 15% of national wealth and income, respectively. While these levels of dispersion mirror those in many countries, there is an underlying downward trend in both income and wealth inequality in Kenya. Specifically, the wealth shares of the richest 1% have been steadily falling, suggesting a slow reduction in wealth inequality. As shown in Figure 1, top 1%'s wealth share falls from a peak of 41% in 2005 to about 28% in 2020.[1] The top 0.1% individuals also experience a similar decline. This trend belies the overall experience of many countries at similar stages of development. In sub-Saharan Africa, wealth inequality is high and rising, with significant variations across countries and regions. According to reports by UNESCO (Adesina, 2016)[2] and the UNDP Regional Bureau for Africa (Cornia, 2017)[3], the average wealth Gini coefficient for the region was 0.65 in 2012, higher than the 0.60 recorded in 1995. However, this masks significant disparities among countries, with South Africa having the highest wealth Gini coefficient of 0.85 in 2012, followed by Namibia (0.80), Angola (0.78), Zambia (0.77), and Nigeria (0.76). On the other hand, some countries have relatively low levels of wealth inequality, such as Ethiopia (0.44), Mali (0.45), Niger (0.46), Burkina Faso (0.47), and Rwanda (0.48). According to the World Bank[4], Kenya's Gini coefficient of wealth inequality fell from 0.81 in 2005 to 0.74 in 2015, while its Gini coefficient of income inequality fell from 0.47 to 0.41 over the same period. These declines are remarkable given that Kenya's per capita income grew by an average of 4.3% per year during this period, which could have increased inequality due to heterogeneous returns to capital and labour (Kuznets, 1955). But the declines in top wealth income

1 Source: Based on WID Data, see Robilliard (2020) for the methodology.

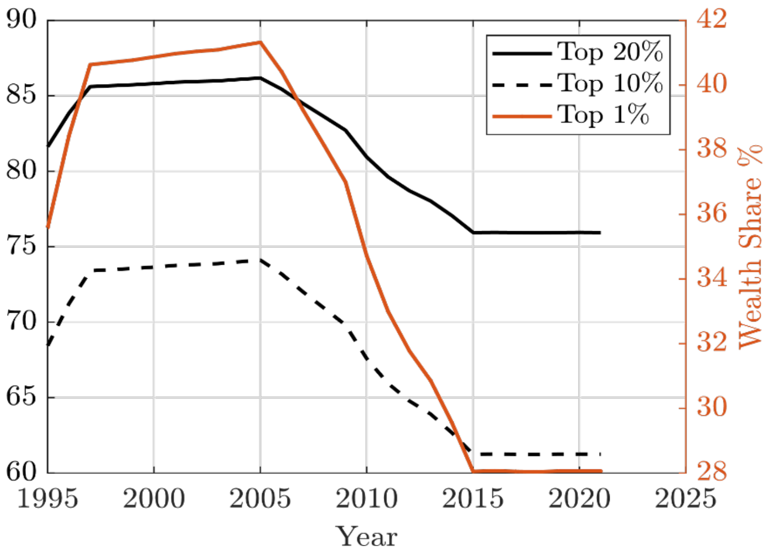
2 Based on data from the Global Wealth Databook 2015 which covers 52 African countries, 47 of which are in sub-Saharan Africa. The Databook is prepared by Credit Suisse (see Credit Suisse, 2015).

3 Based on data from the Integrated Inequality Dataset (IID-SSA). The IID-SSA data set is derived from three sources: WIDER's WIIDv3, The World Bank's Povcal database, and Branko Milanovic's All the Ginis data set. See Cornia and Martorano (2016) for more details.

4 World Development Indicators, 2020.

group shares are also consistent with earlier survey evidence. For example, Suri et al. (2008) used survey data covering a 10-year period between 1997 and 2007, and found that inequality had broadly decreased. Furthermore, they found that reduced inequality was a key driver for the fall in poverty levels in the country, even as real per capita incomes were falling over the period; incomes for the richest 20% fell while those for the bottom 30% rose.

Figure 1: Wealth shares of the top 0.1% and 1% individuals in Kenya



Based on these trends in the data, a key question is what drives the changes in top shares, and why do they substantially differ from observed trends in other countries. This is important since there are contradictory explanations offered in the literature. For example, Bigsten et al. (2014) suggest "changes in income inequality have been strongly influenced by the long-term process of structural change" (p. 8), which has led to the rise in real wages. But Gakuru and Mathenga (2012) uses a social accounting matrix (SAM) to show that inequality in the ownership of factors of production are translated into inequalities in household income, suggesting wage growth is not the main determinant of falling inequality.

We use an Aiyagari–Bewley–Huggett (ABH) class heterogeneous agent macroeconomic model to shed light on these recent trends. ABH models, originally proposed by Bewley (1986), have been used to study representative agent economies with incomplete markets. While not as common as the DSGE macroeconomic models, there is now a well-developed literature that analyses economies with both aggregate uncertainty and a large number of heterogeneous agents (Algan et al., 2014). These models are designed to study whether macroeconomic fluctuations affect different agents differently and in turn, how heterogeneity affects macroeconomic fluctuations. These models have been further refined by the work of Aiyagari (1994) and Krusell and

Smith (1998), and have been used to understand, for example, the increasing wealth of the top 1% net-worth individuals (households) in the United States (see, e.g., Gabaix et al, 2016). Such models are, however, seldom developed to match the moments of the wealth distribution in African countries, despite some of these countries having the highest levels of wealth inequality observed in the world.

This study calibrates an Aiyagari–Bewley–Huggett (ABH) class heterogeneous agent macroeconomic model for the Kenyan economy. The model is then used to recover the empirical distribution of wealth and used to study how fiscal policies affect the top wealth shares. Using data from the Penn World Tables to measure returns to wealth (capital) and tax rates computed based on revenue estimates from the OECD database, we find that our model can account for up to 92% of the variation in top wealth inequality (measured by the share of wealth held by the top 10%), over the 19-year period between 2001 and 2019.[5] We attribute this decline in inequality to three main factors: a persistent but slow increase in the return to capital, a low risk-free rate, and rising effective income tax rates.

The key to this result is the existence of risky and risk-free assets in our version of the ABH heterogeneous agent economy. The risky asset is in the form of productive capital with a stochastic return. Because there are two assets, agents face a portfolio choice problem similar to Merton (1969). With the upward trending return to risky capital, coupled with a low risk-free rate, wealthy individuals allocate a larger share of their net worth to risky investments. This has two effects on wealth inequality. On one hand, a higher volatility of the risky asset increases inequality in that more randomness in the risky asset's return leads to more dispersion in income and subsequently wealth. But on the other hand, higher volatility in the return to capital leads risk-averse individuals to invest a larger share of their wealth in the low-return risk-free asset. This lowers top wealth inequality. The latter effect always dominates, so long as the risk-free rate is lower than the rate of time preference. Progressive taxes have a similar effect—by lowering the effective rate of return of the risky asset, taxation directly reduces inequality. A second channel is that progressive taxation, through its effect on lowering the effective variance of returns to risky investment, again lowers income and, consequently, wealth inequality.

Our study contributes to two strands of literature. First, we contribute to the theoretical literature on heterogeneous agent models with incomplete markets and progressive taxation. Most existing models in this literature assume either a constant or an exogenous stochastic return to capital (e.g., Aiyagari, 1994; Bewley, 1986; Hugget, 1993; Krusell & Smith, 1998). We depart from this assumption by allowing for an endogenous return to capital that depends on the aggregate capital stock in a neoclassical production function. This allows us to capture the dynamic effects of economic growth on inequality through changes in factor prices. Second,

5 This result is based on a regression of the model implied wealth shares and “actual” wealth shares from the World Inequality Database (WID). Note that for the intervening non-survey years, the WID wealth shares estimates are interpolated based on the methodology of Alvaredo, Chancel, Piketty, Saez and Zucman (2020).

we contribute to the policy literature on tax reforms in developing countries. Most existing studies on this topic focus on the revenue and efficiency effects of tax reforms, while paying less attention to their distributional effects (e.g., Zee et al., 2002; Keen et al., 2019). We contribute to filling this gap by showing, analytically, how a tax reform affects, not only the allocation of wealth between risky capital and bonds, but also wealth inequality across different segments of the population.

The rest of the paper is organized as follows. Section 2 describes the data sources and methods that we use to measure income and wealth inequality in Kenya. Section 3 presents our heterogeneous agent model and its implications for top wealth inequality. Section 4 discusses our calibration strategy and its main results on explaining the decline in wealth inequality between 2005 and 2015. Section 5 concludes the study.

2. Data

Survey evidence on inequality in Kenya

Africa has some of the world's most unequal countries but has received little focus in global debate about economic inequality (Nilsson et al., 2023). This lack of attention from the research community may be partly attributable to the difficulty of obtaining long–term cross nationally comparable estimates of income and wealth dispersion. For the overall continent-wide experience, Chancel et al. (2019) provide updated estimates of inequality in Africa from 1990 to 2017 using survey, tax, and national accounts data. Their findings suggest persistently high inequality in the continent: the top 10% of country populations share of income ranges from 36% in Algeria to 67% in Botswana. These levels of inequality stand at par with Latin America or India, with Southern and Central Africa being particularly unequal. They show that the bulk of continent-wide income inequality comes from the within country component, while the between country component has slightly reduced in the two last decades, due to higher growth in poorer countries. They attribute these levels of inequality to dualism between agriculture and other sectors, and mining rents.

For Kenya, the World Inequality Database (WID) provides a similar picture. The WID data includes 399 variables, covering almost all measures of economic activity and wellbeing, over a 25-year time period, 1995 to 2020. For variables such as personal income, wages, wealth and debt, the data include both the average raw value and the share of different population groups. For example, the variable "sptinc992j" that is plotted in Figure 3, is described in the WID metadata as pre-tax national income ... "is the sum of all pre-tax personal income flows accruing to the owners of the production factors labour and capital before taking into account the operation of the tax/transfer system but after taking into account the operation of pension system." The wealth measure is 'net personal wealth', the variable 'ahweali992i', which is described in WID as "the total value of non-financial and financial assets (housing, land, deposits, bonds, equities, etc.) held by households, minus their debts." These data are compiled based on surveys carried out in different countries (Robilliard, 2020).

In the Kenyan case, the data are based on four surveys listed in Table 1. All the surveys are national, with the welfare type (measure) used being consumption. These survey data, which are available from the World Bank's Poverty Inequality Platform

(PIP)[6] or Povcalnet project, are then used to determine the distribution of national income (from National Accounts) in the population. Starting from the Povcalnet information on the distribution of consumption, four adjustments are made in order to estimate the distribution of national income (Chancel et al., 2019). First, since all the surveys use consumption as the welfare measure, the data available are only informative about the distribution of consumption rather than income. In order to obtain the income dispersion, the consumption distribution at each quantile p is multiplied by a consumption–income profile function $c(p)$ to recover the income distribution. Second, the average income of top earners is adjusted to account for two issues: one, the fact that top earners are under-represented in household surveys; and two, that incomes collected in surveys usually correspond to disposable incomes rather than pre-tax incomes. This involves "merging" survey and anonymized tax data, then "reweighting" and "replacing" some data points while retaining the microdata structure of the original survey (Blanchet et al., 2022a). Third, estimates of unearned income components such as taxes on production received by the government, and the retained earnings of corporations are added to the pre-tax income from the previous stage. The gap between this and the national income as recorded in national accounts is then filled by rescaling all incomes to match the national income average. Finally, to recover the full distribution of income, a non-parametric method due to Blanchet et al. (2022b) is used to produce smooth and realistic shapes of generalized Pareto curves. This gives the income distributions for the survey years.

Table 1: Survey information

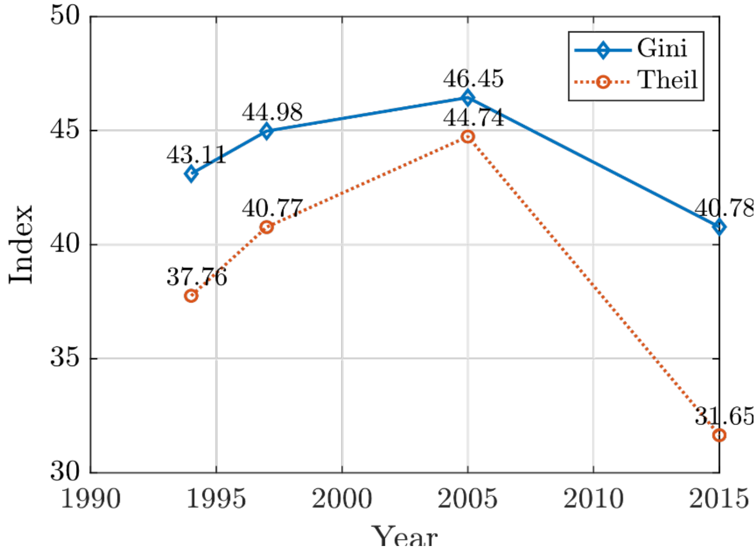
Year	Survey Title	Conductor	Data Type
1992	Welfare Monitoring Survey 1992	CBS	Group
1994	Welfare Monitoring Survey 1994	KNBS	Micro
1997	Welfare Monitoring Survey 1997	KNBS	Micro
2005	Integrated Household Budget Survey 2005–2006	KNBS	Micro
2015	Integrated Household Budget Survey 2015–2016	KNBS	Micro

Notes: CBS is the Central Bureau of Statistics and KNBS is the Kenya National Bureau of Statistics.

Since survey data are only available every five to six years while macroeconomic data comes out annually, the WID database estimates of distribution in non-survey years using interpolation between survey data points. For this reason, we will focus our analysis on the survey years, which still show the general time variation of inequality measures. Based on these data, the World Bank's Poverty Inequality Platform has computed trends in income and wealth dispersion between 1992 and 2015. The Gini and Theil inequality indices, displayed in Figure 2, show the evolution of inequality between 1994 and 2015. As is clearly visible, inequality increases over the first three surveys then sharply falls over the ten-year period between 2005 and 2015.

https://pip.worldbank.org/home

Figure 2: Inequality trend (1994–2015)

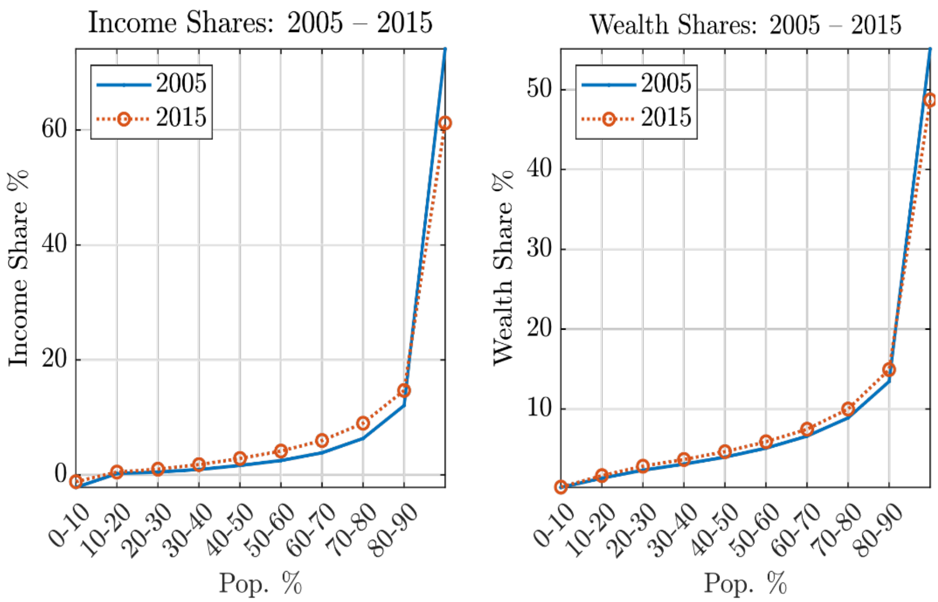


Notes: Based on World Bank PIP data. Graph shows changes in inequality over time according to two indices of inequality: Gini and Theil. Gaps in the time series represent a break in the comparability between two surveys or data points, due to changes in methodology or other factors.

This is further visible in Figure 3, which shows the income and wealth shares of the poorest to the richest deciles of the population over the 10-year period between 2005 and 2015. There is a modest yet clear gain for each decile of the population except the top 10%. This applies to both income and wealth. The gains of the bottom 90% appear to be at the expense of the top 10% who experience a decline in their shares of income. This is consistent with the fall in overall inequality as displayed in Figure 2. The poverty rate consequently falls from 36.67% in 2005 to 29.37% in 2015 as per World Bank estimations. This is consistent with the "observation" that the income and wealth distributions of 2005 are second order stochastically dominant over the 2015 densities (see e.g. Zheng, 2021).[7]

7 For second-degree normalized stochastic dominance (SD), the weighted area between the SD curve of a distribution and that of the equalized distribution is a decomposable inequality measure. SD provides graphical representations for decomposable inequality and poverty measures similar to what the Lorenz curve does for the Gini index.

Figure 3: Evolution of income and wealth shares between 2005 and 2015



Notes: Data from World Inequality Database – computed based on the KNBS Integrated Household Budgets Surveys 2005–2006 and 2015–2016. Each point along the horizontal axis is the respective population decile.

A further look at the data over the survey years shows a rise in inequality between 1995 and 2005, which is more than reversed in the subsequent ten-year period. As can be seen in Table 2, at the right tail of the distribution, the top 1% share of both wealth and income experiences its largest fall between 2005 and 2015. The income shares of the top 1% decline by six percentage points while their wealth share falls by a staggering 13 percentage points. The top 10% experience similar declines, while the bottom 10% experience only modest gains in wealth and income. This suggests that most of the gains in wealth and income lost by the rich accrue to the "middle class", as seen in the left panel of Figure 3 where the bulk of the shift in the distribution occurs between the 40th and 90th deciles. This evolution of income and wealth is contrary to what has been observed in many advanced countries like the United States where the top 0.1% hold 20% of the economy wide net worth.

Table 2: Wealth and income shares over time

	Wealth			Income		
Year	Top 1	Top 10	Bottom 10	Top 1	Top 10	Bottom 10
1995	35.59%	68.44%	−1.58%	19.00%	52.29%	0.18%
1997	40.63%	73.41%	−2.09%	19.06%	54.56%	0.17%
2005	41.32%	74.11%	−2.18%	21.47%	55.11%	0.17%
2015	28.05%	61.24%	−1.18%	15.19%	48.72%	0.21%

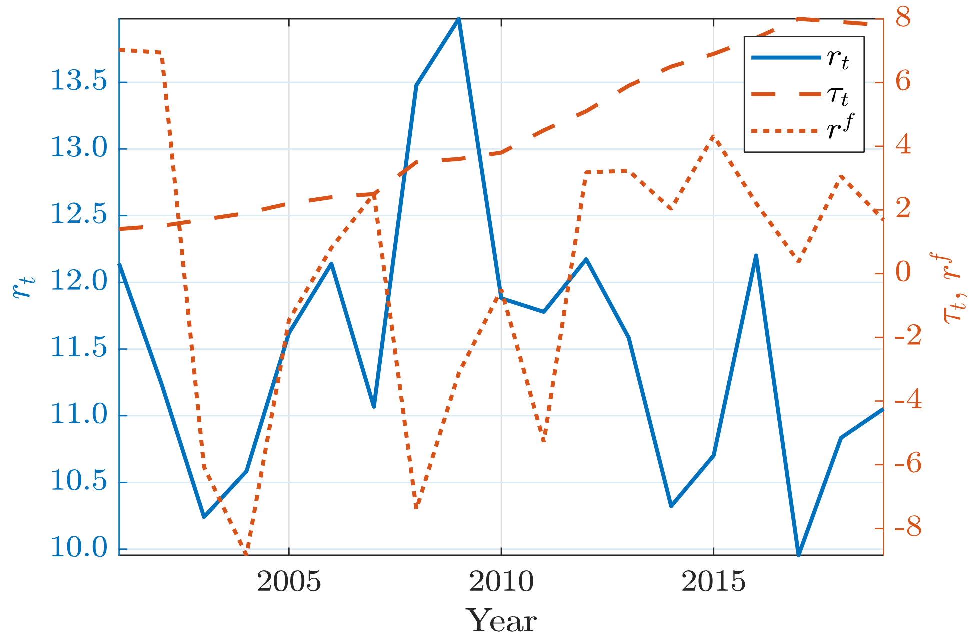
National accounts

One of the objectives of the WID is to use survey data from the World Bank's PIP – Povcalnet project in order to determine the distribution of national income in the population (Alvaredo et al., 2020). National income is defined as gross domestic product, minus consumption of fixed capital, plus net foreign income. The overall objective is to be able to produce Distributional National Accounts (DINA) which provides a link between the macro-level structure of national income and wealth, and on the micro-level distribution of income and wealth. Given this structure of the data, an explanation for the evolution of the distribution of income must take account of the sources of this income.

One method of measuring GDP is the income approach, which can be simplified to accounting for the compensation to factors of production. Assuming national output is generated by a Cobb-Douglas technology $Y_t = X_t K_t^{\alpha} L_t^{1-\alpha}$, where X_t is productivity, K_t is capital, and L_t is labour, then we can write GDP as $Y_t = r_t K_t + w_t L_t$, where r_t and w_t are the returns to capital and wages, respectively. For top wealth inequality, the labour income is negligible so the return to capital is what matters. The share of national income paid to owners of capital is rK/Y. Piketty and Zucman (2015) have argued that because capital income is far more concentrated than labour income, inequality is likely to increase when the capital output ration K/Y rises.

Estimates of the return to capital may be obtained by simply subtracting the compensation of employees from measured gross national income and dividing by the capital stock. But as has been noted by Inklaar et al. (2019), this would lead to large overestimations of the returns to capital as in some countries the rents from extracting natural resources like oil and gas is a sizeable fraction of GDP. So instead we use the rate of return series of the latest version of the Penn World Tables (Feenstra et al.,, 2015) which corrects for this problem. In general (worldwide), the return to capital shows a slowly downward trend. However, in the Kenyan case, the "internal rate of return" series computed by Inklaar et al. (2019) show a slow upward trend with an initial value (mean) of 11.94%, the return gaining around 0.03% per year since 1995 with variations in-between. The return series also potentially show a mean reverting behaviour. The returns are displayed in Figure 4, together with the tax-to-GDP ratio and the risk-free rate (annual Treasury bill rate).

Figure 4: Returns $\mathbf{r_t}$, tax revenue as % of GDP $\boldsymbol{\tau_t}$, and the risk-free rate $\mathbf{r^f}$



Notes: Returns are the IRR obtained from the Penn World Tables; Tax data are from the OECD Global Revenue Statistics Database (Taxes on Income, Profits and Capital Gains, as % of GDP). The risk-free rate is from IMF International Financial Statistics Database.

The tax revenue begins to rise from the year 2002, coinciding with reforms made at the Kenya Revenue Authority (Tyce,2020). This improvement in revenues was mostly driven by a sharp rise in the income tax component of taxes which peaked at 8% of GDP in 2014. This rise in taxes, which is largely driven by an increase in personal income tax, points an origin of the decline in inequality levels.[8] As noted by Moore et al. (2018), the share of *direct taxes* (such as taxes on income, property, and other assets) within a country's overall tax structure can serve as an "equity oriented performance measure" for revenue generation, as "broadly speaking higher dependence on direct taxes indicates a more progressive tax system, in which people with income and

8 The tax disaggregation shows that most of the gains are from taxes on income and profits of individuals and the Pay As You Earn (PAYE) program. The Taxes on Rental Income column is has non-zero entries beginning 2016 suggesting the gains in tax revenue are not driven by taxes on land and/or property rents.

assets pay more than the poor and those without property". Another feature of the data is that the *real* risk-free rate falls sharply from around 7% in 2001 to -8.8% in 2004 (the nominal rates are 12.73% and 2.96%, respectively). The real risk-free rate subsequently never exceeds 4% for the following 20 years. We believe this is partly driven by the rise in government revenue which allowed the Central Bank of Kenya to obtain lower Treasury Bill rates. These two facts, rising revenues and falling risk-free rate, are ultimately the key to understanding the fall in inequality in the subsequent 10-year period.

In the next section, we outline a macroeconomic model that uses these features of the data to account for the decline in wealth and income inequality in Kenya. The key mechanism is that, as the risk-free rate remains persistently low, high net worth individuals are forced to invest a larger share of their wealth in risky projects which are inherently *"less"* risky with a higher taxation.

3. Model economy

Setup

Agents

Consider an economy populated with a continuum of unit mass infinitely lived households. Each household consists of one agent. Household $i \in (0,1)$ has preferences over consumption c_t, defined by:

$$U_0 \equiv \mathbb{E} \int_0^{\infty} e^{-\rho t} u(c_t) dt \tag{1}$$

Where: ρ is the subjective discount factor and the utility function $u(\cdot)$ is defined by

$$u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma}, \quad \gamma \neq 1 \tag{2}$$

Where: $\gamma > 0$ is the coefficient of relative risk aversion. The household is endowed with a unit of time l_t that it inelastically supplies to firms in return to a wage w_t. The household can also own two types of assets: physical capital k_t for which it receives r^J a stochastic return r_t, and riskless bonds b_t which receive a risk-free interest rate .

C Firms

Production in the economy is organized by firms that combine inputs capital k_t and labour l_t to produce output with the production function

$$y_t = X_t k_t^{\alpha} l_t^{1-\alpha},$$

Where: X_t is the economy wide productivity common to all firms. The gross profit of a firm controlling k_t units of capital is: $r_t k_t = y_t - w_t l_t$. Maximizing gross profit of a firm with k_t units of capital, its labour demand will be $l_t = k_t[(1-\alpha)X_t/w_t]^{\frac{1}{\alpha}}$. Given the number of agents in the economy is of unit mass each with a unit of time inelastically supplied to firms, the aggregate labour supply equals unity. If the aggregate capital in the economy equals K_t, then $1 = K_t[(1-\alpha)X_t/w_t]^{\frac{1}{\alpha}}$. Substituting the equilibrium wage w_t back into the expression for gross profit means $r_t = \alpha X_t K_t^{\alpha-1}$. An agent who owns k_t units of physical capital receives the gross return $r_t k_t$.

Return process

We assume that in discrete time, the return r_t is a random walk with a drift:

$$r_t = \mu + r_{t-1} + \epsilon_t \tag{3}$$

Where: $\epsilon_t \sim N(0,\sigma_\epsilon^2)\epsilon_t \sim N(0,\sigma_\epsilon^2)$. Let $t_i = it/nt_i = it/n$ for some natural numbers $i = 0, \dots, ni = 0, \dots, n$. Over the small time step $\Delta t = t_{i+1} - t_i = t/n$ $\Delta t = t_{i+1} - t_i = t/n$, (2) can be written as:

$$r_t \Delta t = (r_0 + \mu t)\Delta t + \sqrt{\Delta t}\sum^{t_n=t} \epsilon_{t_i} \tag{4}$$

with $\mathrm{Var}\left(\sum_{t_i=0}^{t_n} \epsilon_{t_i}\right) = t\sigma_\epsilon^2$. Dividing the ϵ_{t_i}s by their standard deviation $\sqrt{t}\sigma_\epsilon$, we obtain the scaled random walk $\sigma_\epsilon\left(\sqrt{n}\right)^{-1}\sum_{t_i=0}^{t_n=t} \epsilon_{t_i}/\sigma_\epsilon$. Taking limits as $\Delta t \downarrow 0$ with the last term in (4) replaced by the scaled random walk, leads to the following expression for the return to capital:

$$r_t dt = \mu_t dt + \sigma_\epsilon dW_t, \tag{5}$$

Where: $\mu_t = r_0 + \mu t$ and $W_t \sim N(0,t)$ is a standard Brownian motion. Equation 5 turns the discrete time random walk with a drift (3) into constant plus time trend model in continuous time.

Wealth process

Denote wealth (assets) by $a_t = k_t + b_t$. Let θ_t denote the fraction of wealth invested in risky capital. Then the return on the portfolio of capital and bonds over the interval dt is given by:

$$dR_t = (1-\theta)r^f dt + \theta[\mu_t dt + \sigma_\epsilon dW_t].$$

The household's wealth then simply evolves as $da_t = a_t dR + (w_t - c_t)dt$ which simplifies to:

$$da_t = \left[w_t + \left(r^f + \theta(\mu_t - r^f)\right)a_t - c_t\right]dt + \theta\sigma_\epsilon a_t dW_t. \tag{6}$$

The household's problem is then to maximize (1) subject to (6) and the borrowing constraint $a_t > \underline{a}$ where $-\infty < \underline{a} < 0$

Equilibrium

Denote the fraction of households with wealth $a_t \leq a$ by the cumulative distribution function (CDF):

$$F(a,t) = \text{Prob}(a_t \leq a) \tag{7}$$

which satisfies: $F\left(\underline{a},t\right) = 0$ since $a_t \geq \underline{a}$ and $\lim_{a\to\infty} G(a,t) = 1$ with density function $f = \partial_a F(a,t)$. Aggregate capital supply in the economy equals total wealth, which is given by:

$$K_t = \int_0^\infty a\, f(a,t)\, da.$$

Aggregate output in the economy is given by: $Y_t = X_t K_t^\alpha L_t^{1-\alpha}$, where $L_t = 1$ is aggregate labour supply.

Stationary equilibrium

The household's optimal consumption-savings decision can be summarized by two equations: a Hamilton-Bellman-Jacobi (HJB) equation for a value function v and a Fokker-Planck (FP) equation for the density of households f. In a stationary equilibrium, the unknown functions v and f and the scalar r satisfies the following system of coupled partial differential equations (PDEs):

$$\rho v(a) = \max_{c,\theta}\left\{u(c) + v'(a)\left[w + \left(r^f + \theta(\mu - r^f)\right)a - c\right] + \frac{1}{2}v''(a)\theta^2\sigma_\epsilon^2 a^2\right\} \tag{8}$$

$$0 = -[s'(a)f(a) + s(a)f'(a)]da - \left[s'(a)f'(a) + \frac{1}{2}s(a)f''(a)\right](da)^2 \tag{9}$$

Where: $f'(x) = \partial_x f(x)$ and $f'' = \partial_x^2 f(x)$. $s(a)$ is the savings policy function, after dropping labour income, $s(a) = \left(r^f + \theta(\mu - r^f)\right)a - c$. The first order conditions with respect to c and θ implies:

$$c(a) = u'^{-1}\left(v'(a)\right), \quad \theta = -\frac{1}{a}\frac{v'(a)}{v''(a)}\frac{\mu - r^f}{\sigma^2}. \tag{10}$$

With CRRA utility (1), the maximization problem (8) is the portfolio allocation problem of Merton (1969). Using guess and verify methods, we find the optimal policy functions:

$$c(a) = \left(\frac{\rho - (1-\gamma)r^f}{\gamma} - \frac{1-\gamma}{2\gamma^2}\left(\frac{\mu - r^f}{\sigma}\right)^2\right)a \tag{11}$$

$$s(a) = \left(\frac{r^f - \rho}{\gamma} + \frac{1+\gamma}{2\gamma^2}\left(\frac{\mu - r^f}{\sigma}\right)^2\right)a \tag{12}$$

$$k(a) = \frac{1}{\gamma}\left(\frac{\mu - r^f}{\sigma^2}\right)a \tag{13}$$

A complete derivation of (11)–(13) is given Appendix C.

The tail of the wealth distribution

Given the policy function for savings (12), we can find a solution to (9) which gives the stationary distribution of wealth as (see Appendix B):

$$f(a) = a^{-\zeta}, \text{ where } \zeta = (1+\gamma) + \frac{2(r^f - \rho)\gamma\sigma^2}{(\mu - r^f)^2}. \tag{14}$$

This is the Pareto distribution of wealth, with the exponent $\zeta > 1$. Given ζ, we can compute the total wealth share of a proportion p of the wealthiest individuals in the economy as:

$$share = p^{1-\frac{1}{\zeta}} \tag{15}$$

For example, if $\zeta = 2$, then the wealthiest 1% own 10% of total income ($p = 0.01$, and $share = 0.01^{1-\frac{1}{2}} = 10\%$). If $\zeta = 3$, then the top 1% share of total wealth falls to 5%, so top wealth inequality is decreasing in ζ . If we assume that $r^f < \rho$, i.e., that the risk-free rate is always less than the rate of time preference, then a rise in r^f leads to an *increase* in wealth inequality, and a rise in the return to risky capital r , also leads to an increase in wealth inequality. Finally, wealth inequality is *decreasing* in the volatility of return to capital σ. This result is counter intuitive for two reasons. On one hand, a higher σ means there is more randomness in the return to capital which leads to higher wealth inequality. On the other hand, if σ increases, then risk averse households allocate a smaller share of their wealth to the higher return risky asset, leading to a fall in wealth inequality. The second effect dominates if we assume implicitly that the demand for risk-free bonds occasioned by a rise in σ leads further to a fall in r^f.

The power-law distribution (14) provides a powerful tool that links features of the economy to the distribution of wealth. In order to find the distribution of income, we need to find a relationship between wealth and income. This may be done by say regression of the log wealth on log income for different percentiles of the population. This would give a functional relationship between wealth and income, defined by a single parameter, say ψ . It is then straightforward to find the distribution of income based on the distribution of wealth.9

9 Consider a continuous random variable X with the density function $f_X(x)$. Let random variable $Y = g(X)$, where $g(\cdot)$ is one-to-one transformation of X onto Y. Then $Y = g(X)$ is a continuous random variable with the density function

Effect of capital income taxes and economic growth

Capital income taxes

The effects of capital income taxation can be examined within our framework. To do so, we introduce a linear capital income tax so the budget constraint (6) becomes (after dropping labour income):

$$da_t = \left[(1-\tau)\left(r^f + \theta(\mu - r^f)\right)a_t + T_t - c_t\right]dt + \theta(1-\tau)\sigma_\varepsilon a_t dW_t, \quad (16)$$

Where: τ is the linear tax on capital income and T_t are lump-sum transfers. We assume that the government balances its budget each period and redistributes revenues from capital income taxation equally to all individuals. It can then be shown that the share of wealth invested in risky capital becomes:

$$\theta = \frac{1}{\gamma(1-\tau)}\left(\frac{\mu_t - r^f}{\sigma_\epsilon}\right)^2, \quad (17)$$

which shows that, in the presence of taxes, agents invest a larger share of their wealth in the risky asset. Taxation directly lowers the effective volatility of the risky asset, from σ_ϵ to $\sqrt{(1-\tau)}\sigma_\epsilon$. It then follows that ζ from (14) now becomes:

$$\zeta' = \frac{2\gamma + (1-\gamma)(1+\tau)}{(1-\tau)} + \frac{2(r^f - \rho)\gamma\sigma_\epsilon^2}{(\mu - r^f)^2}. \quad (18)$$

Equation (18) shows that taxes reduce wealth inequality, as a higher τ raises ζ which lowers the share of a given top group.

Effect of growth

The model so far has not factored in GDP or wage growth. We assume that labour income w_t grows deterministically at rate growth rate g, then labour income evolves as $w_t = w_t e^{g\tau}$. This is the result of GDP growing at rate g while assuming that the share of labour in aggregate output is a constant: $\hat{w}_t = (1-\alpha)Y_t$. With this assumption at hand, wealth a_t in Equation 6 is non-stationary because w_t is growing.

Define detrended wealth as $\hat{a}_t = e^{-gt}a_t$. We then have: $\frac{d\hat{a}_t}{\hat{a}_t} = \frac{da_t}{a_t} - gdt$. Detrended wealth then follows the process:

$$d\hat{a}_t = [w_t + (r^f + \theta(\mu_t - r^f) - g)\hat{a}_t - c_t]dt + \theta\sigma_\epsilon\hat{a}_t dW_t.$$

$$\hat{\zeta} = (1+\gamma) + \frac{2(r^f - \rho)\gamma\sigma^2}{(\mu - g - r^f)^2},$$ ds to the tail parameter:

(19)

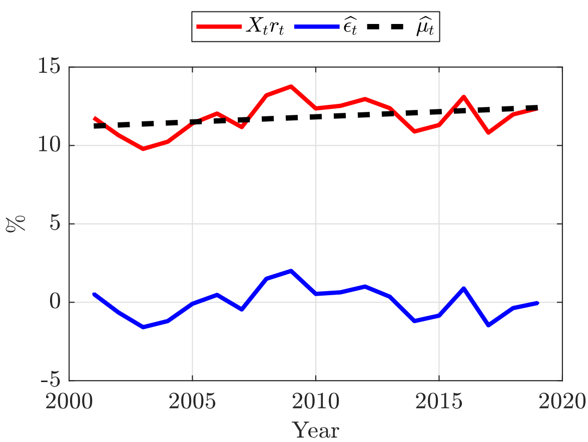
so growth reduces inequality as in Piketty and Zucman's $\bar{r} > g$ hypothesis, where $\bar{r}$ is the the net-of-tax rate of return on wealth and g is the economy's growth rate; with $\bar{r} = (1-\tau)\mu$ in our case.

4. Calibration and results

Baseline

We use data from national accounts to calibrate our model. For the return r_t and total factor productivity X_t, we use data from the Penn World Tables version 9.1 (Feenstra et al., 2015). We use the newly created variable called the "internal rate of return" which is computed using national accounts data as nominal GDP minus labour income minus natural resource rents. The (nominal) internal rate of return on capital is then determined to ensure capital compensation adds up to total capital income (Inklaar et al., 2019). The return enters as $\mu_t = r_0 + \mu t$. To estimate μ_t, we run a regression of $r_t X_t$ on a constant and a linear time trend. This results into the estimates $\widehat{r_0} = 11.24\%$ and $\hat{\mu} = 0.06\%$. We use these two parameters to obtain the errors $\widehat{\epsilon_t} = r_t - \widehat{\mu_t}$, and compute σ_ϵ as the standard deviation of the series $\widehat{\epsilon_t}$. However, to match the model implied wealth shares with data requires setting σ_ϵ to about 25% per annum. The series $r_t X_t$ are displayed Figure 5.

Figure 5: Return to capital



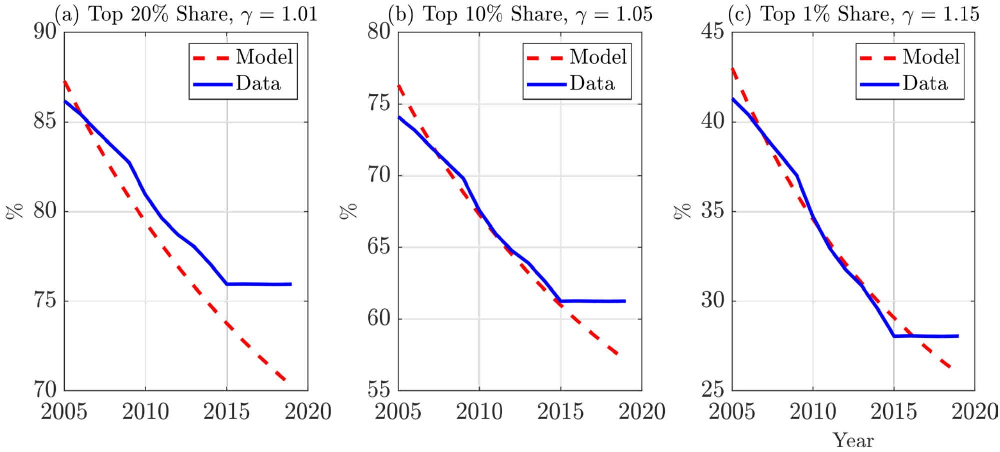
For the risk-free rate r^f, we use the mean of the annual Central Bank of Kenya Treasury Bill Rate, which is sourced from the IMF's International Financial Statistics database. To calibrate the rate of time preference ρ , we use the average real lending interest rate over the preceding 25-year period. Since $e^{-\rho} = (1 + i_t)^{-1}$ $e^{-\rho} = (1 + i_t)^{-1}$, so $\rho = -\log(1 + i_t)^{-1} = 0.0793$. This value just about matches the one used by studies such as Kaplan et al. (2018). Finally, we set the risk aversion parameter to the values of $\gamma = [1.01,1.05,1.25]$ when determining the model implied wealth shares of the top 20%, 10%, and 1% respectively. This is done in order to obtain a closer fit between the model implied wealth shares and also to reflect the fact that the very rich top 1% may have a higher appetite for risk than the less affluent 10% and 20%. While some of the literature set the risk aversion parameter to 2, the values we use are closer to estimates suggested by Chetty (2006) who used labour supply elasticity estimates to show that γ should be approximately equal to unity. Given the above parameters, we find that the model can generally account for the evolution of wealth distribution between the years 2005 and 2015 where survey evidence shows a decline in income and wealth inequality.

Table 3: Wealth shares – model vs. data

Year	Top 20%		Top 10%		Top 1%	
	Model	Data	Model	Data	Model	Data
2005	87.3%	86.2%	76.3%	74.1%	43.0%	41.3%
2015	73.8%	75.9%	61.0%	61.2%	29.1%	28.1%

We summarize these results in Table 3 and Figure 6 (from 2001 to 2019,). As can be seen from Table 3, the model implied wealth share of the richest 1% is very close to the data from 2005. This applies to the top 10% and 20% wealth shares as well. For the 19-year period from 2001 to 2019, the correlation between the model implied wealth shares and the data is at least 93%. Similarly, a regression of the model implied wealth shares on the data gives an R^2 of 89%, suggesting that the model can explain up to 90% of the variation in wealth shares.

Figure 6: Evolution of wealth shares in model vs. data



Notes: The figure plots the share of wealth for various population groups in the economy between 2005 and 2015. The "data" in the intervening years of 2006–2014 are "interpolated" based on the World Bank's PIP projections.

Adding taxes

We use effective tax revenue from the OECD and real GDP data from the World Bank to compute effective tax rates and calibrate the model with capital income tax.[10] However, the tax data begin from 2001, so we limit our analysis to this time period. Figure 7 and Table 4 show the evolution of wealth shares for the top 20%, 10%, and 1%. In order to make the model implied wealth shares closer to the data, we change the values of γ to 1.01 and 2.03 for the top 10% and 1%, respectively. The remaining parameters remain unchanged and are as in the baseline calibration.

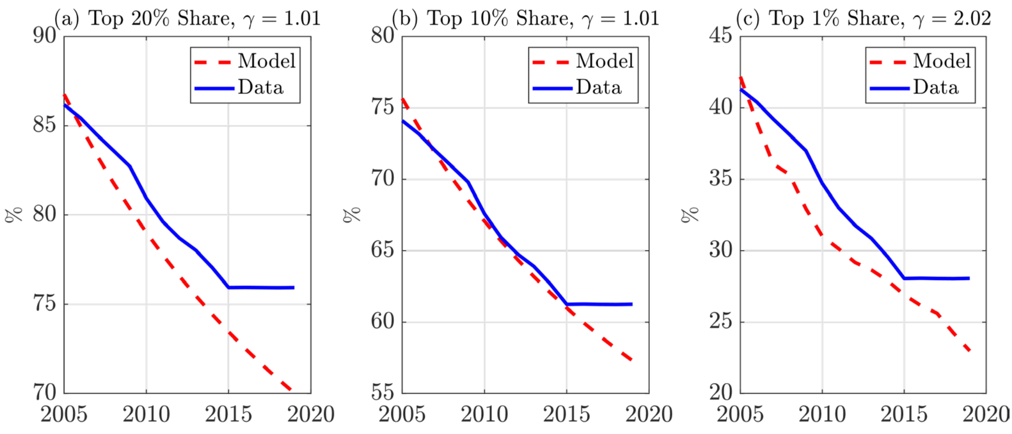
Table 4: Wealth shares with capital income taxes - model vs. data

Year	Top 20%		Top 10%		Top 1%	
	Model	Data	Model	Data	Model	Data
2005	86.8%	86.2%	75.7%	74.1%	42.2%	41.3%
2015	73.5%	75.9%	61.0%	61.2%	26.9%	28.1%

10 Tax Revenue data is from the OECD Global Revenue Statistics Database https://stats.oecd.org/Index.aspx?DataSetCode=REVKEN and the real GDP (in constant national currency units) time series is from https://data.worldbank.org/indicator/NY.GDP.MKTP.KN?locations=KE

The inclusion of taxes in the model leads to a better fit with the data for top 20% and 10% shares. The model implied wealth shares for all groups is much closer to the data in 2005, than in the baseline calibration. In 2015, only the top 20% share has a slightly worse fit. The correlation between the model implied wealth shares and the data is still 94% for the first two groups, but falls slightly for the top 1% to 89%. However, in all cases, a regression with the model wealth shares on the shares from the data still gives an $R^2 = 89\%$, which is equivalent to the model without taxes. While these are similar to the baseline calibration, the fit in the actual survey years suggest that taxes can help explain the decline in top wealth shares. The better fit for the top 20% and top 10% is also consistent with the fact that most of the gains in the effective tax rates (tax-to-GDP ratio) were from increases in income taxes rather than on land and real estate rents.

Figure 7: Evolution of wealth shares in model vs. data □ with capital income taxes



Notes: The figure plots the share of wealth for various population groups in the economy between 2005 and 2015. The “data” in the intervening years of 2006–2014 are “interpolated” based on the World Banks PIP projections.

5. Conclusion

This article set out to explain the recent fall in top wealth and income shares observed in the Kenyan economy. Unlike most major economies, wealth and income inequality has actually been falling in Kenya, particularly over the 10-year period beginning from the year 2005 and ending in 2015. Using a modified version of the heterogeneous agent models due to Aiyagari (1994) and further refined by Achdou et al. (2021), we find that a slowly trending return to wealth coupled with a rising income tax to GDP ratio can explain up to 92% of the variation in wealth shares of different population groups in Kenya. While the model calibration occasionally uses small values of the risk aversion parameter, 1.01 to 1.15 rather than the 1.5 to 2 commonly used in the literature, we are able to account for the evolution of wealth to a high degree of accuracy.

From a policy perspective, the main message of our paper is that lowering wealth inequality requires maintaining an effective fiscal state. This is mainly for two reasons. First, the key driving force to our result is the risk-free rate. It begins a dramatic decline from around the year 2002, a period that coincides with reforms made at the Kenya Revenue Authority. These reforms are subsequently followed by a steady rise in tax revenues (Tyce, 2020). When tax revenues are high enough to cover government expenditures, the risk-free Treasury bill rate falls as government borrowing through the Treasury bond market also falls. This leads investors or capital owners to allocate a larger share of their wealth to riskier productive capital activities. A policy mix of generating enough tax revenue while limiting excessive government borrowing can therefore be an effective tool in the effort to lower both income and wealth inequality.

Finally, there is a real risk that the gains in reducing inequality may be lost due to changes in the tax systems that have recently come into effect over the last two years. While these reforms have led to a reduction in income dispersion, the long-term prognosis is that they will lead to an increase in wealth concentration and income dispersion.

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Appendixes

Appendix A. Household optimization problem

The household's optimization problem is:

$$\max_{c_t,\theta_t} \quad \int_0^{\infty} e^{-\rho t}\, u(c_t)dt \quad s.t.$$

$$da_t \quad = \left[w_t + \left(r^f + \theta(\mu_t - r^f)\right)a_t - c_t\right]dt + \theta\sigma_\epsilon a_t dW_t,$$

$$a_t \geq \underline{a}, \quad \text{with} \quad W_t \sim N(0,t).$$

(A1)

The household chooses a consumption path to maximize her inter-temporal utility subject to the laws of motion for wealth. Denoting the value function by $v(a)$, then over a small time interval Δt , the Bellman equation for the household's prc $v(a_t) = \max_{c_t,\theta}\{u(c_t)\Delta t + e^{-\rho\Delta t}\mathbb{E}_t[v(a_{t+\Delta t})]\}.$

Approximating the discount factor as $e^{-\rho\Delta t} \approx 1 - \rho\Delta t$, we can rewrite (A1) as:

$0 = \max_{c_t,\theta}\{u(c_t)\Delta t + (1-\rho\Delta t)\mathbb{E}_t[v(a_{t+\Delta t})] - v(a_t)\}$

Rearranging and dividing through by Δt gives:

$$\frac{1}{\Delta t}\cdot 0 \quad = \max_{c_t}\left\{u(c_t)\frac{\Delta t}{\Delta t} + \mathbb{E}_t\left[\frac{v(a_{t+\Delta t}) - v(a_t)}{\Delta t}\right] - \rho\frac{\Delta t}{\Delta t}\mathbb{E}_t[v(a_{t+\Delta t})]\right\}.$$

Taking limits as $\Delta t \downarrow 0 \Delta t \downarrow 0$ leads to the expression:

$$\rho v(a_t) = \max_{c_t,\theta}\left\{u(c_t) + \mathbb{E}_t\left[\frac{dv(a_t)}{dt}\right]\right\}.$$

(A2)

We now need to determine the value process $dv(a)$. Since a_t is a stochastic differential equation, we apply Itô's lemma, and have:

$$dv(a) \quad = \left\{v'(a)\left[w_t + \left(r^f + \theta(\mu_t - r^f)\right)a_t - c_t\right] + \frac{1}{2}v''(a)\sigma_\epsilon^2 a_t^2\right\}dt + v'(a)\sigma_\epsilon a_t dW_t.$$

Taking expectations and noting that dW_t is an increment of Brownian motion so that $\mathbb{E}_t[dW_t] = 0$, we have:

$$\frac{1}{dt}\mathbb{E}_t[dv] = v'(a)\left[w_t + \left(r^f + \theta(\mu_t - r^f)\right)a_t - c_t\right] + \frac{1}{2}v''(a)\theta^2\sigma_\epsilon^2 a_t^2.$$

Plugging into (A2) leads to the HJB equation:

$$\rho v(a_t) = \max_{c_t,\theta}\left\{u(c_t) + v'(a)\left[w_t + \left(r^f + \theta(\mu_t - r^f)\right)a_t - c_t\right] + \frac{1}{2}v''(a)\theta^2\sigma_\epsilon^2 a_t^2\right\}$$

(A3)

with the first order conditions: $u'(c_t) = v'(a_t)$ which implies

$$c(a) = u'^{-1}(v'(a)), \text{ and } \theta = -\left(\frac{v'(a)}{av''(a)}\right)\left(\frac{\mu - r^f}{\sigma_\epsilon^2}\right).$$

In the presence of capital income tax, the HJB equation (A3) becomes:

$$\rho v(a_t) = \max_{c_t,\theta}\left\{u(c_t) + v'(a)\left[w_t + (1-\tau)\left(r^f + \theta(\mu_t - r^f)\right)a_t + T_t - c_t\right] + \frac{1}{2}v''(a)(1-\tau)^2\theta^2\sigma_\epsilon^2 a_t^2\right\}$$

$$\rho v(a_t) = \max_{c_t,\theta}\left\{u(c_t) + v'(a)\left[w_t + (1-\tau)\left(r^f + \theta(\mu_t - r^f)\right)a_t + T_t - c_t\right] + \frac{1}{2}v''(a)(1-\tau)^2\theta^2\sigma_\epsilon^2 a_t^2\right\}$$

with the first order conditions with respect to cc and θ becoming:

$$c(a) = u'^{-1}(v'(a)), \text{ and } \theta = -\left(\frac{v'(a)}{av''(a)}\right)\frac{1}{1-\tau}\left(\frac{\mu - r^f}{\sigma_\epsilon^2}\right).$$

Appendix B. Wealth distribution

It is convenient to work with the cumulative distribution function (CDF): $F(a,t)$ $F(a,t)$ which is the fraction of people with wealth $\leq a$

$$F(a,t) = \text{Prob}(a_t \leq a)$$

which satisfies: $F(\underline{a},t) = 0$ since $a_t \geq a$ and $\lim_{a\to\infty} F(a,t) = 1$ $\lim_{a\to\infty} F(a,t) = 1$ with density function $f = \partial_a F(a,t)$. To determine law of motion for F, consider the wealth accumulation process with saving function $s(a) = w + Ra - c$. Consider the proportion of individuals with wealth below a . Over the small time interval Δt for an individual who dis-saves such that $s(a) \leq 0$, we have $a_{t+\Delta t} = a_t - \Delta t \cdot s(a_t)$, and the law of motion for F can then be determined by:

$$\begin{aligned} \text{Prob}(a_{t+\Delta t} < a) &= \underbrace{\text{Prob}(a_t < a)}_{\text{already below } a} + \underbrace{\text{Prob}\big(a \leq a_t \leq a_t - \Delta t \cdot s(a)\big)}_{\text{passes threshold } a} \\ &= \text{Prob}(a \leq a_t \leq a_t - \Delta t \cdot s) \\ F(a, t + \Delta t) &= F(a - \Delta t \cdot s, t) \end{aligned}$$

Subtracting $F(a,t)$ from both sides and dividing by Δt gives, defining $x = \Delta t \cdot s$ and taking limits as $x \to 0$ leads to the Forward Kolmogorov equation:

$$\partial_t F(a,t) = -\partial_a [F(a,t) \cdot s(a)] \tag{B1}$$

In a stationary equilibrium, time is constant $t = 0$ and we have

$\partial_t F(a,0) = \partial_t F(a) = 0$. Equation A4 then becomes $0 = -\partial_a[s(a)F(a)]$. Taking derivative with respect to a and applying Itô lemma, we have:

$$0 \quad = -[s'(a)f(a) + s(a)f'(a)]da - \left[s'(a)f'(a) + \frac{1}{2}s(a)f''(a)\right](da)^2$$

which is the KF Equation 9.

Appendix C. Analytical solutions to the HJB and KF equations

HJB Equation

With the CRRA utility function $u(c_t)$, it is possible to find solutions for the optimal savings and consumption decisions. A standard approach is to guess a functional form for the value function $v(a)$. We guess that $v(a) = Ba^{1-\gamma}$, from which we determine the optimal consumption using first order conditions as $c(a) = [(1-\gamma)B]^{-\frac{1}{\gamma}}a$. Plugging into the HJB Equation (A3) with

$$\theta = \frac{1}{\gamma}\left(\frac{\mu - r^f}{\sigma_\epsilon^2}\right), \quad v'^{(a)} = (1-\gamma)Ba^{-\gamma}, \quad v''^{(a)} = -\gamma(1-\gamma)Ba^{-\gamma-1}$$

while setting labour income to zero and dividing by $Ba^{1-\gamma}$, we have that:

$$\rho = [(1-\gamma)B]^{-\frac{1}{\gamma}} + \left[r^f + \frac{1}{\gamma}\left(\frac{\mu - r^f}{\sigma_\epsilon}\right)^2 - [(1-\gamma)B]^{-\frac{1}{\gamma}}\right](1-\gamma) - \frac{1}{2}\frac{1-\gamma}{\gamma}\left(\frac{\mu - r^f}{\sigma_\epsilon}\right)^2.$$

From which we get an expression for $[(1-\gamma)B]^{-\frac{1}{\gamma}}$. We consequently obtain the optimal consumption, saving, and capital policies (with no labour income and $s(a) = \left(r^f + \theta(\mu - r^f)\right)a - c(a)$, $k(a) = \theta a$) as:

$$\begin{aligned} c(a) &= \left(\frac{\rho-(1-\gamma)r^f}{\gamma} - \frac{1-\gamma}{2\gamma^2}\left(\frac{\mu-r^f}{\sigma_\epsilon}\right)^2\right)a, \\ s(a) &= \left(\frac{r^f-\rho}{\gamma} + \frac{1+\gamma}{2\gamma^2}\left(\frac{\mu-r^f}{\sigma_\epsilon}\right)^2\right)a, \quad \text{and} \\ k(a) &= \frac{1}{\gamma}\left(\frac{\mu-r^f}{\sigma_\epsilon^2}\right)a. \end{aligned}$$

In the presence of capital income taxation with $v(a) = Ba^{(1-\gamma)}$, we have $c(a) = [(1-\gamma)B]^{\ \gamma}a$,

$$\theta = \frac{1}{\gamma(1-\tau)}\left(\frac{\mu-r^f}{\sigma_\epsilon^2}\right) \quad \text{and} \quad T_t = \tau\left(r^f + \theta(\mu_t - r^f)\right)a_t.$$

Plugging into the HJB equation with taxes, we find the following expression for the consumption coefficient:

$$[(1-\gamma)B]^{-\frac{1}{\gamma}} = \frac{\rho}{\gamma} - \frac{1-\gamma}{\gamma}\left(r^f + \frac{(1+\tau)}{2\gamma(1-\tau)}\left(\frac{\mu-r^f}{\sigma_\epsilon}\right)^2\right),$$

which gives the savings function:

$$s(a) = \left(\frac{r^f-\rho}{\gamma} + \frac{2\gamma+(1-\gamma)(1+\tau)}{2\gamma^2(1-\tau)}\left(\frac{\mu-r^f}{\sigma_\epsilon}\right)^2\right)a.$$

KF Equation

Again, we drop labour income from the wealth process (6) which can now be written as:

$$da_t = s(a_t)dt + \theta\sigma_\varepsilon a_t dW_t.$$

Taking the expatiations $\mathbb{E}_t[da_t]$, $\mathbb{E}_t[(da_t)^2]$, dividing by dt and using the multiplication rules for Itô processes, we get:

$$\frac{1}{dt}\mathbb{E}_t[da_t] = s(a_t) = \phi a_t \quad \text{and} \quad \frac{1}{dt}\mathbb{E}_t[(da_t)^2] = \frac{1}{\gamma^2}\left(\frac{\mu_t - r^f}{\sigma_\varepsilon}\right)^2 a_t^2 = \psi a_t^2.$$

Dropping the time subscripts and using $s'(a) = \phi$ and plugging into Equation 9, we have:

$$0 \quad = -\phi^2[f(a) + af'(a)]a - \phi\psi\left[f'(a) + \frac{1}{2}af''(a)\right]a^2 \tag{C1}$$

which is an "Euler Equations" type ordinary differential equation with a known solution of the form $f(a) = a^q$ for $a > 0$ with the exponent q to be determined. Plugging into (C1) with $f'(a) = qa^{q-1}$ and $f''(a) = q(q-1)a^{q-2}$ and simplifying, we obtain the quadratic equation:

$$-(q+1)\left(\phi + \frac{1}{2}\psi q\right), \text{with roots} \quad q = -1,\ -2\frac{\phi}{\psi}.$$

Choosing the second quadratic root, we consequently have an expression for the stationary distribution of wealth:

$$f(a) = a^{-\zeta}, \text{where } \zeta = (1+\gamma) + \frac{2(r^f - \rho)\gamma\sigma_\varepsilon^2}{(\mu - r^f)^2}.$$

In the presence of capital income taxes, we have from the savings function that,

$$\phi = \left(\frac{r^f - \rho}{\gamma} + \frac{2\gamma + (1-\gamma)(1+\tau)}{2\gamma^2(1-\tau)}\left(\frac{\mu - r^f}{\sigma_\epsilon}\right)^2\right.$$

while ψ is as before, then the exponent ζ for the wealth distribution becomes:

$$\zeta' = \frac{2\gamma + (1-\gamma)(1+\tau)}{(1-\tau)} + \frac{2(r^f - \rho)\gamma\sigma_\epsilon^2}{(\mu r - r^f)^2},$$

which equals ζ when τ is set to zero.

Mission

To strengthen local capacity for conducting independent, rigorous inquiry into the problems facing the management of economies in sub-Saharan Africa.

The mission rests on two basic premises: that development is more likely to occur where there is sustained sound management of the economy, and that such management is more likely to happen where there is an active, well-informed group of locally based professional economists to conduct policy-relevant research.

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